16-09-2020

B.Sc. Physics Part-III

Paper-V, Group-A

Three dimensional Laplace's equation in Cartesian coordinate:

Laplace's equation $\nabla^2 u = 0$

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{—— (1)}$$

Let the solution of (1) is given by the following form.

$$u = X(x)Y(y)Z(z) \quad \text{—— (2)}$$

Substituting eqn-(2) in eqn-(1), we obtain

$$Y(y)Z(z)\frac{\partial^2 X}{\partial x^2} + X(x)Z(z)\frac{\partial^2 Y}{\partial y^2} + X(x)Y(y)\frac{\partial^2 Z}{\partial z^2} = 0$$

dividing by XYZ,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

or $$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0 \quad \text{—— (3)}$$

Let us take

$$\frac{1}{X}\frac{d^2 X}{dx^2} = k^2$$

$$\frac{1}{Y}\frac{d^2 Y}{dy^2} = l^2$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -(k^2 + l^2)$$

$\{ k^2, l^2$ are constant

or

$$\frac{d^2 X}{dx^2} - k^2 X = 0; \quad \frac{d^2 Y}{dy^2} - l^2 Y = 0$$

and $$\frac{d^2 Z}{dz^2} + (k^2 + l^2) Z = 0$$

Solution of above equations are given by

$$X = a_1 e^{kx} + a_2 e^{-kx}$$

$$Y = a_3 e^{ly} + a_4 e^{-ly}$$

$$Z = a_5 \cos\sqrt{(k^2+l^2)}\, z + a_6 \sin\sqrt{(k^2+l^2)}\, z$$

Therefore, the possible solution of Laplace's equation in Cartesian coordinate is given by

$$\boxed{u = (a_1 e^{kx} + a_2 e^{-kx})(a_3 e^{ly} + a_4 e^{-ly})(a_5 \cos\sqrt{k^2+l^2}\, z + a_6 \sin\sqrt{k^2+l^2}\, z)}$$

We can also take constants as $-k^2, -l^2$ and (k^2+l^2). In this case solution is given by

$$\boxed{u = (a_1 \cos kx + a_2 \sin kx)(a_3 \cos ly + a_4 \cos ly)(a_5 e^{\sqrt{k^2+l^2}\, z} + a_6 e^{-\sqrt{k^2+l^2}\, z})}$$